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THE LINEAR DIFFERENCE EQUATION OF THE FIRST ORDER.*

By K. P. Williams.

In this paper we consider the first order linear difference equation

$$f(x+1) = E(x)f(x), \tag{A}$$

where $E(x)$ is an entire function. It will be seen that the theory for such an equation gives at once the theory for the equation where the coefficient of $f(x)$ is a meromorphic function.

The equation (A) was first completely solved by Hurwitz,† who showed the existence of a solution when $E(x)$ is any entire function. He did not, however, consider this equation primarily, but rather the equation

$$f(x+1) - f(x) = \varphi(x), \tag{B}$$

for which he proved that there was an entire solution in case $\varphi(x)$ is entire,‡ and a meromorphic solution in case $\varphi(x)$ is meromorphic. The logarithmic derivative of the first equation reduces to one of the form (B), where the known function is meromorphic. Thus the solution Hurwitz gave of (B) included that of our equation.

Somewhat later Barnes§ gave a direct treatment of equation (A), by expressing the function $E(x)$ in its Weierstrass factor form. To include all cases he found it necessary, however, to know that equation (B) has a solution when $\varphi(x)$ is an entire function. But Hurwitz proved in an elegant way that to pass from the case where $\varphi(x)$ in (B) is entire to the case where it is meromorphic, is but a simple step, and this, as remarked above, is to know completely the theory for (A). Thus we gain nothing if we appeal at all to (B) in giving a direct solution for the first equation. The solution that Barnes obtained is, furthermore, to a large extent undetermined; for it contains certain functions of the zeros of $E(x)$, whose explicit forms, he states, he was able to determine only in certain simple cases.

The present discussion also depends fundamentally on the factor form of $E(x)$; but the method is essentially different from that used by Barnes. The solution which is obtained is simple, and is completely determined.

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† Acta Mathematica (1897), T. XX, pp. 285–312.

‡ An interesting discussion of (B) for the case where $\varphi(x)$ is entire is given by Carmichael, Amer. Journal of Math., vol. XXXV, pp. 165–182.

§ Proceedings of London Math. Soc. (1904), 2d series, vol. 2, pp. 438–469. Barnes considers the equation $f(x+\omega) = \varphi(x)f(x)$, where ω is any real or complex constant. This is no more general, and adds only to the complexity of the formulæ.

We restrict ourselves entirely to those cases which do not need to be made to depend on (B); this includes all cases where $E(x)$ is an entire function of finite genus, and also a large class of cases where this limitation is not fulfilled. It is also thought that the present method has the advantage of showing clearly why a simple choice of certain convergence factors can be made.

Let a_1, a_2, $\cdots$, be the zeros of $E(x)$; then we know by Weierstrass's factor theorem that we can always write

$$E(x) = e^{g(x)} \prod_{i=1}^{\infty} \left(1 - \frac{x}{a_i}\right) e^{\sum_{n=1}^{i-1} \frac{1}{n} \left(\frac{x}{a_i}\right)^n},$$

where $g(x)$ is itself some entire function. If we can find an integer k such that the series

$$\sum_{i=1}^{\infty} \left| \frac{1}{a_i} \right|^{k+1}$$

is convergent, we say the function $E(x)$ is of finite order, and we have the simpler form

$$E(x) = e^{g(x)} \prod_{i=1}^{\infty} \left(1 - \frac{x}{a_i}\right) e^{\sum_{n=1}^{k} \frac{1}{n} \left(\frac{x}{a_i}\right)^n}.$$

If further $g(x)$ reduces to a polynomial of degree q, we say that the function $E(x)$ is of finite genus r, where r is the greatest of k and q, provided k is the smallest positive integer for which the above series is convergent. We shall limit ourselves to the cases where $g(x)$ is a polynomial, the order being either finite or infinite.

The resolution of $E(x)$ into factors suggests at once a method for solving our equation; for a fundamental property of such an equation is that the solution of

$$f(x + 1) = u_1(x)u_2(x)f(x),$$

where $u_1(x)$ and $u_2(x)$ are known, is found by forming the product of the solutions of the two equations

$$f(x + 1) = u_1(x)f(x), \quad \text{and} \quad f(x + 1) = u_2(x)f(x).$$

In order to be able to write the solutions corresponding to the exponential factors in $E(x)$ we note first that a solution of the simple equation

$$f(x + 1) = e^{ax^n} f(x),$$

where a is any constant, and n a positive integer, is given immediately by

$$f(x) = e^{\frac{a\phi_{n+1}(x)}{n+1}},$$

where $\varphi_{n+1}(x)$ is the nth Bernoulli polynomial. The fundamental principle mentioned above then shows at once how to solve the equation $f(x+1) = e^{v(x)}f(x)$, where $v(x)$ is any polynomial. In the cases we are considering we can thus solve easily the equation

$$f(x+1) = e^{g(x)}f(x),$$

which arises from the extraneous exponential factor in $E(x)$; we indicate the solution by $e^{\bar{g}(x)}$, $\bar{g}(x)$ being a polynomial which can be explicitly expressed in terms of the Bernoulli polynomials and the constants in $g(x)$.

If now we assume that equation (B) has a solution whenever $\varphi(x)$ is entire, we see that the equation which we have last written has a solution even in case $g(x)$ ceases to be a polynomial, but is any undetermined entire function. It was precisely at this point that Barnes introduced (B) into his consideration of (A).

We notice finally that a solution of the equation $f(x+1) = (1 - x/a_i)f(x)$, is given by $f(x) = \Gamma(x - a_i)(-a_i)^{-x}$, $\Gamma(x)$ being the gamma function. We thus have solutions corresponding to all the types of factors in $E(x)$; and from these we build up a solution of (A).

Let us first assume that the function $E(x)$ is of finite order. The equation then takes the form

$$f(x+1) = f(x)e^{g(x)} \prod_{i=1}^{\infty} \left(1 - \frac{x}{a_i}\right) e^{\sum_{n=1}^{k} \frac{1}{n}\left(\frac{x}{a_i}\right)^n},$$

the series $\sum_{i=1}^{\infty} \left|\frac{1}{a_i}\right|^{k+1}$ being convergent.

A formal solution is at once seen to be

$$f(x) = e^{\bar{g}(x)} \prod_{i=1}^{\infty} \frac{\Gamma(x - a_i)e^{\theta_i(x)}}{(-a_i)^x p_i},$$

where

$$\theta_i(x) = \frac{\varphi_2(x)}{1 \cdot 2a_i} + \frac{\varphi_3(x)}{2 \cdot 3a_i^2} + \cdots + \frac{\varphi_{k+1}(x)}{k(k+1)a_i^k},$$

while $p_1, p_2, \cdots$ are any constants (or periodic functions of period 1).

If now we can choose the quantities p_i so that the above expression is a uniformly convergent product we shall have an analytic solution of our equation. A method for doing this will be found by means of the asymptotic form of the gamma function.

Before we proceed, however, we shall introduce a certain convention which will save unnecessary repetition. We have frequent occasion to use functions of x and a_i, say $M(x, a_i)$, which have the property of being less in absolute value than some properly chosen constant, provided x is confined to any limited region, and $|a_i|$ is large enough. Since $|a_i|$ increases indefinitely with i, the last condition will be fulfilled if i is sufficiently great, say if $i > I$. The value of I depends in general on the region in which x can vary. Under these conditions we shall merely say that $M(x, a_i)$ is bounded, *always* understanding by that term that x and i satisfy the requisite conditions; that is, x is in some limited region, while i is larger than some positive integer, depending perhaps on the region in which x can vary. Similarly, when we say a function of a_i alone is bounded, we shall *always* mean that it is less in absolute value than some constant if $|a_i|$ is sufficiently great, that is, if we choose i large enough. Such functions as those described will occur in our work upon breaking off convergent series, or series which asymptotically represent a function.

In order to be able to expand $\Gamma(x - a_i)$, we shall assume for the present that all the points $a_1, a_2, \cdots$ lie to the left of a certain line parallel to the imaginary axis, and are arranged in order of increasing absolute values. We can then use Stirling's formula for the gamma function, and we thus obtain

$$\Gamma(x - a_i) = \sqrt{2\pi}(x - a_i)^{x-a_i-\frac{1}{2}}e^{-x+a_i+\psi(x-a_i)},$$

where

$$\psi(x - a_i) = \frac{B_1}{1 \cdot 2(x - a_i)} - \frac{B_2}{3 \cdot 4(x - a_i)^3} + \cdots + \frac{(-1)^{t+1}B_t}{(2t - 1)2t(x - a_i)^{2t-1}} + \frac{R_t(x, a_i)}{(x - a_i)^{2t+1}},$$

the B's being the Bernoulli numbers, t any positive integer, and $R_t(x, a_i)$ a bounded function.

Therefore if i is sufficiently great we have for the general term of our formal product

$$\frac{\Gamma(x - a_i)e^{\theta_i(x)}}{(-a_i)^x p_i} = \frac{\sqrt{2\pi}(-a_i)^{-a_i-\frac{1}{2}}}{p_i}e^{\omega_i(x)},$$

where, for short, we have put

$$\omega_i(x) = a_i - x + \left(x - a_i - \frac{1}{2}\right)\log\left(1 - \frac{x}{a_i}\right) + \theta_i(x) + \psi(x - a_i).$$

We next expand $\omega_i(x)$ in terms of powers of $1/a_i$. In the first place we have

$$-x + \left(x - a_i - \frac{1}{2}\right)\log\left(1 - \frac{x}{a_i}\right)$$
$$= -\frac{x^2 - x}{1 \cdot 2a_i} - \frac{\left(x^3 - \frac{3}{2}x^2\right)}{2 \cdot 3a_i^2} - \cdots - \frac{\left(x^{k+1} - \frac{k+1}{2}x^k\right)}{k(k+1)a_i^k} + \frac{\eta_k(x, a_i)}{a_i^{k+1}},$$

where $\eta(x, a_i)$ is bounded.

The function $\theta_i(x)$ is already a polynomial in $1/a_i$, so that we need merely transform $\psi(x - a_i)$.

To find the desired expression for $\psi(x - a_i)$ let us formally expand the divergent infinite series

$$\frac{B_1}{1 \cdot 2(x - a_i)} - \frac{B_2}{3 \cdot 4(x - a_i)^3} + \frac{B_3}{5 \cdot 6(x - a_i)^5} - \cdots,$$

as a series in $1/a_i$. We obtain for the typical term the expression

$$\frac{1}{s(s+1)a_i^s}\left\{ -\binom{s+1}{2}B_1x^{s-1} + \binom{s+1}{4}B_2x^{s-3} + \cdots \right\},$$

where the quantity in parentheses ends with $(-1)^{(s+1)/2}B_{(s+1)/2}$, if s is odd, and in the term in x if s is even. We can simplify this result by making use of the Bernoulli polynomials.

We know that

$$\varphi_n(x) = x^n - \frac{n}{2}x^{n-1} + \binom{n}{2}B_1x^{n-2} - \binom{n}{4}B_2x^{n-4} + \cdots,$$

the expression terminating with the term in x or x^2. The general term of the above expansion can thus be written

$$\frac{1}{s(s+1)a_i^s}\left\{ x^{s+1} - \frac{s+1}{2}x^s - \varphi_{s+1}(x) + (-1)^{\frac{s+1}{2}}B_{\frac{s+1}{2}} \right\},$$

if we agree to put $B_{(s+1)/2}$ equal to zero when s is even.

Let us next choose the arbitrary integer t which occurs in $\psi(x - a_i)$ so that $2t + 1 \geqq k + 1$. It follows then, by making use of the expression last obtained, that we can write

$$\begin{aligned}\psi(x - a_i) &= \frac{1}{1 \cdot 2a_i}\{x^2 - x - \varphi_2(x) - B_1\} + \frac{1}{2 \cdot 3a_i^2}\left\{ x^3 - \frac{3}{2}x^2 - \varphi_3(x) \right\} \\ &\quad + \cdots + \frac{1}{k(k+1)a_i^k}\left\{ x^{k+1} - \frac{k+1}{2}x^k - \varphi_{k+1}(x) + (-1)^{\frac{k+1}{2}}B_{\frac{k+1}{2}} \right\} \\ &\quad + \frac{R(x, a_i)}{a_i^{k+1}},\end{aligned}$$

where $R(x, a_i)$ is bounded.

We now have all the quantities which occur in $\omega_i(x)$ written in terms of $1/a_i$. Adding them we find the simple result

$$\omega_i(x) = a_i - \frac{B_1}{1 \cdot 2a_i} + \frac{B_2}{3 \cdot 4a_i^3} - \frac{B_3}{5 \cdot 6a_i^5} + \cdots + \frac{(-1)^{\frac{k+1}{2}}B_{\frac{k+1}{2}}}{k(k+1)a_i^k} + \frac{M(x, a_i)}{a_i^{k+1}},$$

where $M(x, a_i) = \eta(x, a_i) + R(x, a_i)$ is bounded.

The essential property of the expression which we have derived for $\omega_i(x)$ *is that the terms involving* $1/a_i$, $1/a_i^3$, $\cdots$, $1/a_i^k$, *do not contain the variable* x, *and are, moreover, the first terms in the series which occurs in the asymptotic form of* $\Gamma(-a_i)$. This fact enables us to determine in a simple manner suitable choices for $p_1, p_2, \cdots$.

Let us put

$$p_i = \Gamma(-a_i) = \sqrt{2\pi}(-a_i)^{-a_i-\frac{1}{2}}e^{\xi(a_i)},$$

where

$$\xi(a_i) = a_i - \frac{B_1}{1\cdot 2a_i} + \frac{B_2}{3\cdot 4a_i^3} - \cdots + \frac{(-1)^{\frac{k+1}{2}}B_{\frac{k+1}{2}}}{k(k+1)a_i^k} + \frac{\epsilon(a_i)}{a_i^{k+1}},$$

$\epsilon(a_i)$ being bounded.

With this selection the typical term of our infinite product is, for large enough i,

$$\frac{\Gamma(x-a_i)e^{\theta_i(x)}}{(-a_i)^x\Gamma(-a_i)} = e^{\frac{M'(x,\,a_i)}{a_i^{k+1}}},$$

where $M'(x, a_i)$ is bounded.

What we have proved is this: *As soon as the region in which x can vary is known we can find a positive integer I, and a positive constant K, such that, if $p_i = \Gamma(-a_i)$, the general term of our product is given by the last formula, in which $|M'(x, a_i)| < K$, provided only that $i > I$.*

But we know that the series

$$\sum_{i=1}^{\infty}\left|\frac{1}{a_i^{k+1}}\right|$$

is convergent, and we thus see at once that the selection we have made for p_i makes our product converge uniformly in any region where its factors are all analytic.

We have then as a solution of the given equation the function

$$f(x) = e^{\bar{g}(x)}\prod_{i=1}^{\infty}\frac{\Gamma(x-a_i)e^{\sum_{n=2}^{k+1}\frac{\phi_n(x)}{(n-1)na_i^{n-1}}}}{(-a_i)^x\Gamma(-a_i)}.$$

This solution is seen to have no zeros in the finite plane, while the nature and situation of its singularities follow at once from the theory of the gamma function.

Let us take next the case where the zeros of $E(x)$ are such that they lie to the right of a line parallel to the axis of imaginaries. We obtain a solution by merely everywhere replacing the gamma function in the above result by the second gamma function; the second gamma function, $\bar{\Gamma}(u)$, being the solution of

$$f(x+1) = xf(x),$$

that is asymptotic to Stirling's series in the left half plane; it is connected with $\Gamma(u)$ by the relation $\bar{\Gamma}(u) = (1 - e^{2\pi u\sqrt{-1}})\Gamma(u)$.

Consider now the case where $E(x)$ is any entire function of finite genus. To apply the results already obtained we divide the zeros into two groups; those on or to the left of the imaginary axis forming one group, and those on the right the other. Let us call these two groups $a_1, a_2, \cdots$, and $b_1, b_2, \cdots$, respectively. The solution is then found in a perfectly obvious way as the product of the solutions of two equations of the types we have considered, and is

$$f(x) = e^{\bar{g}(x)} \prod_{i=1}^{\infty} \frac{\Gamma(x - a_i) e^{\sum_{n=2}^{k+1} \frac{\phi_n(x)}{(n-1)n a_i^{n-1}}}}{(-a_i)^x \Gamma(-a_i)} \prod_{i=1}^{\infty} \frac{\bar{\Gamma}(x - b_i) e^{\sum_{n=2}^{k+1} \frac{\phi_n(x)}{(n-1)n b_i^{n-1}}}}{(-b_i)^x \bar{\Gamma}(-b_i)}.$$

Since the second gamma function, $\bar{\Gamma}(u)$, is zero for $u = 1, 2, \cdots$, our solution is a meromorphic function, with its zeros at the points

$$x = b_i + n \qquad (i, n = 1, 2, \cdots),$$

and its poles at the points

$$x = a_i - n \quad (i = 1, 2, \cdots, n = 0, 1, 2, \cdots).$$

So far we have assumed explicitly that $E(x)$ is of finite order. Let us now suppose that this is no longer the case, but that the function $g(x)$ is still a polynomial. By making a modification of our reasoning exactly similar to that for finding the Weierstrass primary factors for the corresponding case we derive a solution, which is obtained from the one above, for the case of finite order, by making the summation which appears as the exponent of e in the ith factor include terms from $n = 2$ to $n = i$.

We shall finally consider briefly the case where the coefficient of $f(x)$ in (A) is a meromorphic function. The results which we have already obtained will enable us to show the existence of a solution provided a certain restriction is fulfilled.

Suppose first that the coefficient is the reciprocal of an entire function. The substitution $f(x) = 1/f(x)$ reduces the equation at once to the equation we have considered. This enables us to treat the general case; for we know that any meromorphic function is the quotient of two entire functions. Let us denote the extraneous exponential factor in these two functions by $e^{g_1(x)}$ and $e^{g_2(x)}$, respectively. Then if $g_1(x) - g_2(x)$ reduces to a polynomial we see that the solution can be written as the quotient of the solutions of two equations, each of which we know can be solved. The solution will be a meromorphic function, with its zeros and poles at known points.

The class of entire and meromorphic functions for which we have obtained a simple and completely determined solution of (A) is seen to be a very wide one.

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